Inequality involving altitudes of a triangle.
https://www.linkedin.com/groups/8313943/8313943-6418854553294237699
In any triangle ABC with usual notions, prove that
(h_a/a)² + (h_b/b)² + (h_c/c)² ≥ 9/4.
Proposed by Vasile Vanciu.
Solution by Arkady Alt , **San Jose**, **California**, **USA**.
Let F area of $\triangle ABC$ and let $\Delta(x,y,z) := 2xy+2yz+zx-x^2-y^2-z^2$
Since $h_a = \dfrac{2F}{a}$ we obtain $\sum\left(\dfrac{h_a}{a}\right)^2 \geq \dfrac{9}{4} \Leftrightarrow 16F^2\sum\dfrac{1}{a^4} \geq 9 \Leftrightarrow$
(**1**) $\quad \Delta(a^2,b^2,c^2)\left(\dfrac{1}{a^4}+\dfrac{1}{b^4}+\dfrac{1}{c^4}\right) \geq 9.$
First note that inequality of the problem isn't holds for any triangle but holds holds for any non obtuse triangle.
Indeed, let $\triangle ABC$ be obtuse isosceles triangle with $C = 120^\circ$ and $AC = BC = 1,$ that is $a = b = 1, c = \sqrt{3}$. Then $F = \dfrac{\sqrt{3}}{4}$ and $16F^2\sum\dfrac{1}{a^4} = 3\left(2+\dfrac{1}{9}\right) = \dfrac{19}{3} < 9.$
Since inequality
(**2**) $\quad \Delta(a,b,c)\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}\right) \geq 9$
holds for any triangle with sidelengths a,b,c then inequality (**1**) holds for any acute angled triangle because in that case a^2,b^2,c^2 satisfies to triangle inequalities.
Inequality (**2**) immediately follows from double inequality

$$\frac{9a^2b^2c^2}{a^2b^2+b^2c^2+c^2a^2} \leq \frac{3abc(a+b+c)}{a^2+b^2+c^2} \leq \Delta(a,b,c) \qquad [1].$$

Indeed, $\dfrac{9a^2b^2c^2}{a^2b^2+b^2c^2+c^2a^2} \leq \Delta(a,b,c) \Leftrightarrow 9 \leq \dfrac{\Delta(a,b,c)(a^2b^2+b^2c^2+c^2a^2)}{a^2b^2c^2} =$
$\Delta(a,b,c)\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}\right).$
Remains to prove that inequality $16F^2\sum\dfrac{1}{a^4} \geq 9$ holds for any right angled triangles.
Let $\triangle ABC$ be right angled triangles with hypotenuse c. Then $F = \dfrac{ab}{2}, c^2 = a^2+b^2$ and
$16F^2\sum\dfrac{1}{a^4} = 4a^2b^2\left(\dfrac{1}{a^4}+\dfrac{1}{b^4}+\dfrac{1}{(a^2+b^2)^2}\right) =$
$4\left(\dfrac{(a^2+b^2)^2}{a^2b^2}-2+\dfrac{a^2b^2}{(a^2+b^2)^2}\right) = 4\left(t+\dfrac{1}{t}-2\right)$, where $t := \dfrac{(a^2+b^2)^2}{a^2b^2} \geq 4.$
Since $t+\dfrac{1}{t}$ increase if $t \geq 1$ then $t+\dfrac{1}{t} \geq 4+\dfrac{1}{4}$ for $t \geq 4$ and, therefore,
$4\left(t+\dfrac{1}{t}-2\right) \geq 4\left(4+\dfrac{1}{4}-2\right) = 9.$
1.Geometric Inequalities with polynomial $2xy+2yz+zx-x^2-y^2-z^2$, **Arkady Alt**, **OCTOGON MATHEMATICAL MAGAZINE vol. 22**,**n.2**,
p.738, More inequality with Δ, after words "So it remains to prove:
1. $\dfrac{3abc(a+b+c)}{a^2+b^2+c^2} \leq \Delta(a,b,c) \leq \dfrac{8abc(ab+bc+ca)}{(a+b)(b+c)(c+a)}$
2. $\dfrac{9a^2b^2c^2}{a^2b^2+b^2c^2+c^2a^2} \leq \dfrac{3abc(a+b+c)}{a^2+b^2+c^2} \Leftrightarrow$

$$3abc(a^2 + b^2 + c^2) \leq (a + b + c)(a^2b^2 + b^2c^2 + c^2a^2).$$ "

Link:
http://www.equationroom.com/Publications/OCTOGON%20Mathematical%20Magazine/Geometric